On the spaces of equivariant maps between real algebraic varieties

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概 要

Recently the author notices that the stability dimension obtained in [1] and [12] can be improved by using the truncated simplicial resolutions invented by J. Mostovoy [15]. The purpose of this note is to announce these improvements.

1 Introduction.

We consider the homotopy types of spaces of algebraic (rational) maps from real projective space $\mathbb{RP}^m$ into the complex projective space $\mathbb{CP}^m$ for $2 \leq m \leq 2n$. It is known in [1] that the inclusion of the space of rational (or regular) maps into the space of all continuous maps is a homotopy equivalence. These results combined with those of [1] can be formulated as a single statement about $\mathbb{Z}/2$-equivariant homotopy equivalence between these spaces, where the $\mathbb{Z}/2$-action is induced by the complex conjugation. This is also one of the generalizations of a theorem of [9], and it is already published in [12]. Recently the author notices that the stability dimensions given in [1] and [12] can be improved by using the truncated simplicial resolutions invented by J. Mostovoy [15]. In this note we shall announce about these improvements (cf. [2]).

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1.1 Definitions and notations.

Let $\mathbb{K}$ denote one of the fields $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers and let $d(\mathbb{K}) = \dim_{\mathbb{R}} \mathbb{K} = 1$ if $\mathbb{K} = \mathbb{R}$ and 2 if $\mathbb{K} = \mathbb{C}$. Let m and n be positive integers such that $1 \leq m < d(\mathbb{K}) \cdot (n+1) - 1$. We choose $\mathbf{e}_m^{\mathbb{K}} = [1 : 0 : \cdots : 0] \in \mathbb{K}\mathrm{P}^m$ as the base point of $\mathbb{K}\mathrm{P}^n$. For $d(\mathbb{K}) \leq m < d(\mathbb{K}) \cdot (n+1) - 1$, we denote by $\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$ the space consisting of all based maps $f : (\mathbb{R}\mathrm{P}^m, \mathbf{e}_m^{\mathbb{R}}) \to (\mathbb{K}\mathrm{P}^n, \mathbf{e}_n^{\mathbb{K}})$, and by $\mathrm{Map}_\epsilon^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$, where $\epsilon \in \mathbb{Z}/2 = \{0, 1\} = \pi_0(\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n))$, the corresponding path component of $\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$. Similarly, let $\mathrm{Map}(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$ denote the space of all free maps $f : \mathbb{R}\mathrm{P}^m \to \mathbb{K}\mathrm{P}^n$ and $\mathrm{Map}_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$ the corresponding path component of $\mathrm{Map}(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$.

We shall use the symbols z_i when we refer to complex valued coordinates or variables or when we refer to complex and real valued ones at the same time while the notation x_i will be restricted to the purely real case.

A map $f : \mathbb{R}\mathrm{P}^m \to \mathbb{K}\mathrm{P}^n$ is called a *algebraic map of the degree* d if it can be represented as a rational map of the form $f = [f_0 : \cdots : f_n]$ such that $f_0, \cdots, f_n \in \mathbb{K}[z_0, \cdots, z_m]$ are homogeneous polynomials of the same degree d with no common *real* roots except $\mathbf{0}_{m+1} = (0, \cdots, 0) \in \mathbb{R}^{m+1}$.

We denote by $\mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$ (resp. $\mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$) the space consisting of all (resp. based) algebraic maps $f : \mathbb{R}\mathrm{P}^m \to \mathbb{K}\mathrm{P}^n$ of degree d. It is easy to see that there are inclusions $\mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n) \subset \mathrm{Map}_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$ and $\mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n) \subset \mathrm{Map}_{[d]_2}^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$, where $[d]_2 \in \mathbb{Z}/2 = \{0, 1\}$ denotes the integer d mod 2. Let $A_d(m, n)(\mathbb{K})$ denote the space consisting of all $(n+1)$-tuples $(f_0, \cdots, f_n) \in \mathbb{K}[z_0, \cdots, z_m]^{n+1}$ of homogeneous polynomials of degree d with coefficients in $\mathbb{K}$ and without non-trivial common real roots (but possibly with non-trivial common *complex* ones).

Let $A_d^{\mathbb{K}}(m, n) \subset A_d(m, n)(\mathbb{K})$ be the subspace consisting of $(n+1)$-tuples $(f_0, \cdots, f_n) \in A_d(m, n)(\mathbb{K})$ such that the coefficient of z_0^d in f_0 is 1 and 0 in the other f_k's $(k \neq 0)$. Then there is a natural surjective projection map

$$\Psi_d^{\mathbb{K}} : A_d^{\mathbb{K}}(m, n) \to \mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n).$$

For $m \geq 2$ and $g \in \mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{K}\mathrm{P}^n)$ a fixed algebraic map, we denote by $\mathrm{Alg}_d^{\mathbb{K}}(m,n;g)$ and $F(m,n;g)$ the spaces defined by

$$\begin{cases} \mathrm{Alg}_d^{\mathbb{K}}(m,n;g) & = \ \{f \in \mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n) : f|\mathbb{R}\mathrm{P}^{m-1} = g\}, \\ F^{\mathbb{K}}(m,n;g) & = \ \{f \in \mathrm{Map}_{[d]_2}^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n) : f|\mathbb{R}\mathrm{P}^{m-1} = g\}. \end{cases}$$

Note that there is a homotopy equivalence $F^{\mathbb{K}}(m,n;g) \simeq \Omega^m \mathbb{K}\mathrm{P}^n$. Let $A_d^{\mathbb{K}}(m,n;g) \subset A_d^{\mathbb{K}}(m,n)$ denote the subspace given by

$$A_d^{\mathbb{K}}(m,n;g) = (\Psi_d^{\mathbb{K}})^{-1}(\mathrm{Alg}_d^{\mathbb{K}}(m,n;g)).$$

Observe that if an algebraic map $f \in \mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$ can be represented as $f = [f_0 : \cdots : f_n]$ for some $(f_0, \cdots, f_n) \in A_d^{\mathbb{K}}(m,n)$ then the same map can also be represented as $f = [\tilde{g}_m f_0 : \cdots : \tilde{g}_m f_n]$, where $\tilde{g}_m = \sum_{k=0}^m z_k^2$. So there is an inclusion

$$\mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n) \subset \mathrm{Alg}_{d+2}^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$$

and we can define *the stabilization map* $s_d : A_d^{\mathbb{K}}(m,n) \to A_{d+2}^{\mathbb{K}}(m,n)$ by $s_d(f_0, \cdots, f_n) = (\tilde{g}_m f_0, \cdots, \tilde{g}_m f_n)$.

It is easy to see that there is a commutative diagram

$$\begin{array}{ccc} A_d^{\mathbb{K}}(m,n) & \xrightarrow{s_d} & A_{d+2}^{\mathbb{K}}(m,n) \\ {\scriptstyle \Psi_d^{\mathbb{K}}}\downarrow & & {\scriptstyle \Psi_{d+2}^{\mathbb{K}}}\downarrow \\ \mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n) & \xrightarrow{\subset} & \mathrm{Alg}_{d+2}^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n) \end{array}$$

A map $f \in \mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$ is called an algebraic map of *minimal degree* d if $f \in \mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n) \setminus \mathrm{Alg}_{d-2}^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$. It is easy to see that if $g \in \mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{K}\mathrm{P}^n)$ is an algebraic map of minimal degree d, then the restriction

$$\Psi_d^{\mathbb{K}}|A_d^{\mathbb{K}}(m,n;g) : A_d^{\mathbb{K}}(m,n;g) \xrightarrow{\cong} \mathrm{Alg}_d^{\mathbb{K}}(m,n;g)$$

is a homeomorphism. Let

$$\begin{cases} i_{d,\mathbb{K}} : \mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n) \xrightarrow{\subset} \mathrm{Map}_{[d]_2}^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n) \\ i'_{d,\mathbb{K}} : \mathrm{Alg}_d^{\mathbb{K}}(m,n;g) \xrightarrow{\subset} F(m,n;g) \simeq \Omega^m \mathbb{K}\mathrm{P}^n \end{cases}$$

denote the inclusions and let

$$i_d^{\mathbb{K}} = i_{d,\mathbb{K}} \circ \Psi_d^{\mathbb{K}} : A_d^{\mathbb{K}}(m,n) \to \mathrm{Map}_{[d]_2}^*(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n).$$

be the natural projection.

1.2 The case $m = 1$.

First, recall the following old result for the case $m = 1$.

Theorem 1.1 ([10], [20] (cf. [13])). *Let $n \geq 2$ and $d \geq 1$ be integers.*

(i) *If $\mathbb{K} = \mathbb{R}$ and $m = 1$, the map $i_d^{\mathbb{R}} : A_d^{\mathbb{R}}(1, n) \to \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^n) \simeq \Omega S^n$ is a homotopy equivalence up to dimension $D_1(d, n)$, where $D_1(d, n)$ denotes the integer given by $D_1(d, n) = (d + 1)(n - 1) - 1$.*

Moreover, if $n \geq 3$ or $n = 2$ with $d \equiv 1 \pmod 2$, there is a homotopy equivalence $A_d^{\mathbb{R}} \simeq J_d(\Omega S^n)$, where $J_d(\Omega S^n)$ denotes the d-th stage James filtration of ΩS^n given by

$$J_d(\Omega S^n) = S^{n-1} \cup e^{2(n-1)} \cup e^{3(n-1)} \cup \cdots \cup e^{d(n-1)} \subset \Omega S^n.$$

(ii) *If $\mathbb{K} = \mathbb{C}$ and $m = 1$, the map $i_d^{\mathbb{C}} : A_d^{\mathbb{C}}(1, n) \to \Omega S^{2n+1}$ is a homotopy equivalence up to dimension $D_1(d, 2n+1) = 2n(d+1) - 1$ and there is a homotopy equivalence $A_d^{\mathbb{C}}(1, n) \simeq J_d(\Omega S^{2n+1})$.*

Remark. (i) A map $f : X \to Y$ is called *a homotopy* (resp. *a homology*) *equivalence up to dimension D* if $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is an isomorphism for any $k < D$ and an epimorphism for $k = D$. Similarly, it is called *a homotopy* (resp. *a homology*) *equivalence through dimension D* if $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is an isomorphism for any $k \leq D$.

(ii) Let G be a finite group and let $f : X \to Y$ be a G-equivariant map. Then a map $f : X \to Y$ is called *a G-equivariant homotopy* (resp. *homology*) *equivalence up to dimension D* if for each subgroup $H \subset G$ the induced homomorphism $f_*^H : \pi_k(X^H) \to \pi_k(Y^H)$ (resp. $f_*^H : H_k(X^H, \mathbb{Z}) \to H_k(Y^H, \mathbb{Z})$) is an isomorphism for any $k < D$ and an epimorphism for $k = D$.

Similarly, it is called *a G-equivariant homotopy* (resp. *homology*) *equivalence through dimension D* if for each subgroup $H \subset G$ the induced homomorphism $f_*^H : \pi_k(X^H) \xrightarrow{\cong} \pi_k(Y^H)$ (resp. $f_*^H : H_k(X^H, \mathbb{Z}) \xrightarrow{\cong} H_k(Y^H, \mathbb{Z})$) is an isomorphism for any $k \leq D$.

The complex conjugation on $\mathbb{C}$ naturally induces the $\mathbb{Z}/2$-action on $A_d^{\mathbb{C}}(m,n)$ and S^{2n+1}, where we identify S^{2n+1} with the space

$$S^{2n+1} = \{(w_0, \cdots, w_n) \in \mathbb{C}^{n+1} : \sum_{k=0}^{n} |w_k|^2 = 1\}.$$

It is easy to see that $A_d^{\mathbb{C}}(m,n)^{\mathbb{Z}/2} = A_d^{\mathbb{R}}(m,n)$ and $(i_d^{\mathbb{C}})^{\mathbb{Z}/2} = i_d^{\mathbb{R}}$. Hence, we also have:

Corollary 1.2 ([10]). *If $n \geq 2$ and $d \geq 1$ are integers, the map $i_d^{\mathbb{C}} : A_d^{\mathbb{C}}(1,n) \to \Omega S^{2n+1}$ is a $\mathbb{Z}/2$-equivariant homotopy equivalence up to dimension $D_1(d,n)$.*

2 The case $m \geq 2$.

2.1 The improvements of the stability dimensions.

For a space X, let $F(X,r)$ denote the configuration space of distinct r points in X given by $F(X,r) = \{(x_1, \cdots, x_r) \in X^r : x_i \neq x_j \text{ if } i \neq j\}$. The symmetric group S_r of r letters acts on $F(X,r)$ freely by permuting coordinates. Let $C_r(X)$ be the configuration space of unordered r-distinct points in X given by the orbit space $C_r(X) = F(X,r)/S_r$.

It is known ([8], [18]) that there are the stable homotopy equivalence and the isomorphism of abelian groups

$$\begin{cases} \Omega^m S^{m+l} \simeq_s \bigvee_{r=1}^{\infty} D_r(\mathbb{R}^m; S^l) & \text{(stable homotopy equivalence)} \\ H_k(D_r(\mathbb{R}^m, S^l), \mathbb{Z}) \cong H_{k-rl}(C_r(\mathbb{R}^m), (\pm\mathbb{Z})^{\otimes r}) & (k, l \geq 1), \end{cases}$$

where we set $\bigwedge^r X = X \wedge \cdots \wedge X$ (r times), $X_+ = X \cup \{*\}$ ($*$ is the disjoint base point), and $D_r(\mathbb{R}^m, S^l) = F(\mathbb{R}^m, r)_+ \wedge_{S_r} \left(\bigwedge^r S^l\right)$.

Let $G_{m,N;k}^M$ and $D_{\mathbb{K}}(d;m,n)$ be the abelian group and the positive integer defined by

$$\begin{cases} G^M_{m,N;k} = \displaystyle\bigoplus_{r=1}^{M} H_{k-(N-m)r}(C_r(\mathbb{R}^m),(\pm\mathbb{Z})^{\otimes(N-m)}), \\ D_{\mathbb{K}}(d;m,n) = \begin{cases} (n-m)\big(\lfloor\frac{d+1}{2}\rfloor+1\big)-1 & \text{if } \mathbb{K}=\mathbb{R},\ d\leq 3, \\ (n-m)d-2 & \text{if } \mathbb{K}=\mathbb{R},\ d\geq 4, \\ (2n-m+1)\big(\lfloor\frac{d+1}{2}\rfloor+1\big)-1 & \text{if } \mathbb{K}=\mathbb{C},\ d\leq 3, \\ (2n-m+1)d-2 & \text{if } \mathbb{K}=\mathbb{C},\ d\geq 4, \end{cases} \end{cases}$$

where $\lfloor x\rfloor$ denotes the integer part of a real number x. Note that there is an isomorphism $H_k(\Omega^m S^{m+l},\mathbb{Z})\cong G^{\infty}_{m,m+l;k}$ for any $k\geq 1$.

Then we have the following results.

Theorem 2.1 (cf. [1]). *Let $2\leq m<n$ and let $g\in \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^{m-1},\mathbb{R}\mathrm{P}^n)$ be an algebraic map of minimal degree d.*

(i) *The inclusion $i'_{d,\mathbb{R}}:\mathrm{Alg}^{\mathbb{R}}_d(m,n;g)\to F^{\mathbb{R}}(m,n;g)\simeq \Omega^m S^n$ is a homotopy equivalence through dimension $D_{\mathbb{R}}(d;m,n)$ if $m+2\leq n$ and a homology equivalence through dimension $D_{\mathbb{R}}(d;m,n)$ if $m+1=n$.*

(ii) *For any $k\geq 1$, $H_k(\mathrm{Alg}^{\mathbb{R}}_d(m,n;g),\mathbb{Z})$ contains the subgroup $G^d_{m,n;k}$ as a direct summand. Moreover, the induced homomorphism $i'_{d,\mathbb{R}*}:H_k(\mathrm{Alg}^{\mathbb{R}}_d(m,n;g),\mathbb{Z})\to H_k(\Omega^m S^n,\mathbb{Z})$ is an epimorphism for any $k\leq (n-m)(d+1)-1$.*

Theorem 2.2 (cf. [1]). *If $2\leq m<n$ are positive integers,*

$$i^{\mathbb{R}}_d : A^{\mathbb{R}}_d(m,n)\to \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m,\mathbb{R}\mathrm{P}^n)$$

is a homotopy equivalence through dimension $D_{\mathbb{R}}(d;m,n)$ if $m+2\leq n$ and a homology equivalence through dimension $D_{\mathbb{R}}(d;m,n)$ if $m+1=n$.

Theorem 2.3 (cf. [12]). *Let $2\leq m\leq 2n$, and let $g\in \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^{m-1},\mathbb{C}\mathrm{P}^n)$ be an algebraic map of minimal degree d.*

(i) *The inclusion $i'_{d,\mathbb{C}}:\mathrm{Alg}^{\mathbb{C}}_d(m,n;g)\to F^{\mathbb{C}}(m,n;g)\simeq \Omega^m S^{2n+1}$ is a homotopy equivalence through dimension $D_{\mathbb{C}}(d;m,n)$ if $m<2n$ and a homology equivalence through dimension $D_{\mathbb{C}}(d;m,n)$ if $m=2n$.*

(ii) *For any* $k \geq 1$, $H_k(\mathrm{Alg}^{\mathbb{C}}_d(m,n;g),\mathbb{Z})$ *contains the subgroup* $G^d_{m,2n+1;k}$ *as a direct summand. Moreover, the induced homomorphism* $i'_{d,\mathbb{C}*}$: $H_k(\mathrm{Alg}^{\mathbb{C}}_d(m,n;g),\mathbb{Z}) \to H_k(\Omega^m S^{2n+1},\mathbb{Z})$ *is an epimorphism for any* $k \leq (2n-m+1)(d+1)-1$.

Theorem 2.4 (cf. [12])**.** *If* $2 \leq m \leq 2n$ *are positive integers,*

$$i^{\mathbb{C}}_d : A^{\mathbb{C}}_d(m,n) \to \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$$

is a homotopy equivalence through dimension $D_{\mathbb{C}}(d;m,n)$ *if* $m < 2n$ *and a homology equivalence through dimension* $D_{\mathbb{C}}(d;m,n)$ *if* $m = 2n$.

Note that the complex conjugation on $\mathbb{C}$ naturally induces $\mathbb{Z}/2$-actions on the spaces $\mathrm{Alg}^{\mathbb{C}}_d(m,n;g)$ and $A^{\mathbb{C}}_d(m,n)$ as before. In the same way it also induces a $\mathbb{Z}/2$-action on $\mathbb{C}\mathrm{P}^n$ and this action extends to actions on the spaces $\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^{2n+1})$ and $\mathrm{Map}^*_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$, where we identify $S^{2n+1} = \{(w_0,\cdots,w_n) \in \mathbb{C}^{n+1} : \sum_{k=0}^n |w_k|^2 = 1\}$ and regard $\mathbb{R}\mathrm{P}^m$ as a $\mathbb{Z}/2$-space with the trivial $\mathbb{Z}/2$-action.

Corollary 2.5 (cf. [12])**.** *Let* $2 \leq m \leq 2n$, $d \geq 1$ *be positive integers and* $g \in \mathrm{Alg}^{\mathbb{C}}_d(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{C}\mathrm{P}^n)$ *be a fixed algebraic map of the minimal degree* d.

(i) *If* $m < 2n$, *the inclusion map* $i'_{d,\mathbb{C}} : \mathrm{Alg}^{\mathbb{C}}_d(m,n;g) \to F^{\mathbb{C}}(m,n;g) \simeq \Omega^m S^{2n+1}$ *is a* $\mathbb{Z}/2$*-equivariant homotopy equivalence through dimension* $D_{\mathbb{R}}(d;m,n)$.

(ii) *If* $m = 2n$, *the above inclusion map* $i'_{d,\mathbb{C}}$ *is and a* $\mathbb{Z}/2$*-equivariant homology equivalence through dimension* $D_{\mathbb{R}}(d;m,n)$.

(iii) *The map* $i^{\mathbb{C}}_d : A^{\mathbb{C}}_d(m,n) \to \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ *is a* $\mathbb{Z}/2$*-equivariant homotopy equivalence through dimension* $D_{\mathbb{R}}(d;m,n)$ *if* $m < 2n$ *and a* $\mathbb{Z}/2$*-equivariant homology equivalence through the same dimension* $D_{\mathbb{R}}(d;m,n)$ *if* $m = 2n$.

2.2 Conjectures.

Finally we report several related questions.

Conjecture 2.6. *Is the projection* $\Psi^{\mathbb{K}}_d : A^{\mathbb{K}}_d(m,n) \to \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{K}\mathrm{P}^n)$ *a homotopy equivalence?*

Let $\hat{D}_{\mathbb{K}}(d;m,n)$ denote the integer given by

$$\hat{D}_{\mathbb{K}}(d;m,n)=\begin{cases}(n-m)(d+1)-1 & \text{if } \mathbb{K}=\mathbb{R},\\ (2n-m+1)(d+1)-1 & \text{if } \mathbb{K}=\mathbb{C}.\end{cases}$$

Conjecture 2.7. *Is the map $i_d^{\mathbb{K}}: A_d^{\mathbb{K}}(m,n)\to \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m,\mathbb{K}\mathrm{P}^n)$ a homotopy (or homology) equivalence up to dimension $\hat{D}_{\mathbb{K}}(d;m,n)$?*

Remark. The above conjectures are correct if $m=1$.

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